

Destroying Non-Complete Regular Components in Graph Partitions

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Abstract

We prove that if G is a graph and $r_1, \dots, r_k \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^k r_i \geq \Delta(G) + 2 - k$ then $V(G)$ can be partitioned into sets V_1, \dots, V_k such that $\Delta(G[V_i]) \leq r_i$ and $G[V_i]$ contains no non-complete r_i -regular components for each $1 \leq i \leq k$. In particular, the vertex set of any graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ sets, each of which induces a disjoint union of triangles and paths.

1 Introduction

In [5] Kostochka modified an algorithm of Catlin to show that every triangle-free graph G can be colored with at most $\frac{2}{3}(\Delta(G) + 3)$ colors. In fact, his modification proves that the vertex set of any triangle-free graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ sets, each of which induces a disjoint union of paths. We generalize this as follows.

Main Lemma. *Let G be a graph and $r_1, \dots, r_k \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^k r_i \geq \Delta(G) + 2 - k$. Then $V(G)$ can be partitioned into sets V_1, \dots, V_k such that $\Delta(G[V_i]) \leq r_i$ and $G[V_i]$ contains no non-complete r_i -regular components for each $1 \leq i \leq k$.*

Setting $k = \left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ and $r_i = 2$ for each i gives a slightly more general form of Kostochka's theorem.

Corollary 1. *The vertex set of any graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ sets, each of which induces a disjoint union of triangles and paths.*

For coloring, this actually gives the bound $\chi(G) \leq 2 \left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ for triangle free graphs. To get $\frac{2}{3}(\Delta(G) + 3)$, just use $r_k = 0$ when $\Delta \equiv 2 \pmod{3}$.

Similarly, for any $r \geq 2$, setting $k = \left\lceil \frac{\Delta(G)+2}{r+1} \right\rceil$ and $r_i = r$ for each i gives the following.

Corollary 2. *Fix $r \geq 2$. The vertex set of any K_{r+1} -free graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{r+1} \right\rceil$ sets each inducing an $(r-1)$ -degenerate subgraph with maximum degree at most r .*

For the purposes of coloring it is more economical to split off $\Delta + 2 - (r+1) \left\lfloor \frac{\Delta+2}{r+1} \right\rfloor$ parts with $r_j = 0$.

Corollary 3. *Fix $r \geq 2$. The vertex set of any K_{r+1} -free graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{r+1} \right\rceil$ sets each inducing an $(r-1)$ -degenerate subgraph with maximum degree at most r and $\Delta(G) + 2 - (r+1) \left\lfloor \frac{\Delta(G)+2}{r+1} \right\rfloor$ independent sets. In particular, $\chi(G) \leq \Delta(G) + 2 - \left\lfloor \frac{\Delta(G)+2}{r+1} \right\rfloor$.*

For $r \geq 3$, the bound on the chromatic number is only interesting in that its proof does not rely on Brooks' theorem. In [7] Lovász proved a decomposition lemma of the same form as the Main Lemma. The Main Lemma gives a more restrictive partition at the cost of replacing $\Delta(G) + 1$ with $\Delta(G) + 2$.

Lovasz's Decomposition Lemma. *Let G be a graph and $r_1, \dots, r_k \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^k r_i \geq \Delta(G) + 1 - k$. Then $V(G)$ can be partitioned into sets V_1, \dots, V_k such that $\Delta(G[V_i]) \leq r_i$ for each $1 \leq i \leq k$.*

For $r \geq 3$, combining this with Brooks' theorem gives the following better bound for a K_{r+1} -free graph G (first proved in [1], [3] and [6]):

$$\chi(G) \leq \Delta(G) + 1 - \left\lfloor \frac{\Delta(G) + 1}{r + 1} \right\rfloor.$$

2 The Proofs

Instead of proving directly that we can destroy all non-complete r -regular components in the partition, we prove the theorem for the more general class of r -permissible graphs and show that non-complete r -regular graphs are r -permissible.

Definition 1. For a graph G and $r \geq 0$, let G^r be the subgraph of G induced on the vertices of degree r in G .

Definition 2. Fix $r \geq 2$. A collection T of graphs is *r -permissible* if it satisfies all of the following conditions.

1. Every $G \in T$ is connected.
2. $\Delta(G) = r$ for each $G \in T$.
3. $\delta(G^r) > 0$ for each $G \in T$.
4. If $G \in T$ and $x \in V(G^r)$, then $G - x \notin T$.
5. If $G \in T$ and $x \in V(G^r)$, then there exists $y \in V(G^r) - N_G(x)$ such that $G - y$ is connected.
6. Let $G \in T$ and $x \in V(G^r)$. Put $H = G - x$. Let $A \subseteq V(H)$ with $|A| = r$. Let y be some new vertex and form H_A by joining y to A . If $H_A \in T$, then $A \cap N_G(x) \cap V(G^r) \neq \emptyset$.

For $r = 0, 1$ the empty set is the only r -permissible collection.

Lemma 4. Fix $r \geq 2$ and let T be the collection of all non-complete connected r -regular graphs. Then T is r -permissible.

Proof. Note that for $G \in T$ we have $G^r = G$. Now (1), (2), (3) and (4) are clearly satisfied. Since each $G \in T$ is non-complete, it has at least two end blocks. Thus if $x \in V(G^r)$ we can always pick some y such that $G - y$ is connected in an end block not containing x . Hence (5) holds. That (6) holds is immediate from regularity. Hence T is r -permissible. \square

Now to prove the Main Lemma we just need to prove the following.

Lemma 5. *Let G be a graph and $r_1, \dots, r_k \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^k r_i \geq \Delta(G) + 2 - k$. If T_i is an r_i -permissible collection for each $1 \leq i \leq k$, then $V(G)$ can be partitioned into sets V_1, \dots, V_k such that $\Delta(G[V_i]) \leq r_i$ and $G[V_i]$ contains no element of T_i as a component for each $1 \leq i \leq k$.*

Proof. For a graph H , let $c(H)$ be the number of components in H and let $p_i(H)$ be the number of components of H that are members of T_i . For a partition $P = (V_1, \dots, V_k)$ of $V(G)$ let

$$\begin{aligned} f(P) &= \sum_{i=1}^k (|E(G[V_i])| - r_i |V_i|), \\ c(P) &= \sum_{i=1}^k c(G[V_i]), \\ p(P) &= \sum_{i=1}^k p_i(G[V_i]). \end{aligned}$$

Let $P = (V_1, \dots, V_k)$ be a partition of $V(G)$ minimizing $f(P)$ and then $c(P)$ and then $p(P)$.

Let $1 \leq i \leq k$ and $x \in V_i$ with $d_{G[V_i]}(x) \geq r_i$. Since $\sum_{i=1}^k r_i \geq \Delta(G) + 1 - k$ there is some $j \neq i$ such that $d_{G[V_j]}(x) \leq r_j$. Moving x from V_i to V_j gives a new partition P^* with $f(P^*) \leq f(P)$. Note that if $d_{G[V_i]}(x) > r_i$ we would have $f(P^*) < f(P)$ contradicting the minimality of P . This proves that $\Delta(G[V_i]) \leq r_i$ for each $1 \leq i \leq k$.

Now assume that for some i_1 there is $A_1 \in T_{i_1}$ which is a component of $G[V_{i_1}]$. Plainly, we may assume that $r_{i_1} \geq 2$. Put $P_1 = P$ and $V_{1,i} = V_i$ for $1 \leq i \leq k$. Take $x_1 \in V(A_1^{r_{i_1}})$ such that $A_1 - x_1$ is connected (this exists by condition (5) of r -permissibility). By the above we have $i_2 \neq i_1$ such that moving x_1 from V_{1,i_1} to V_{1,i_2} gives a new partition $P_2 = (V_{2,1}, V_{2,2}, \dots, V_{2,k})$ such that $f(P_2) = f(P_1)$. By the minimality of $c(P_1)$, x_1 is adjacent to only one component C_2 in $G[V_{2,i_2}]$. Let $A_2 = G[V(C_2) \cup \{x_1\}]$. Since (by condition (4)) we destroyed a T_{i_2} component when we moved x_1 out of V_{1,i_1} , by the minimality of $p(P_1)$, it must be that $A_2 \in T_{i_2}$. Now pick $x_2 \in A_2^{r_{i_2}}$ not adjacent to x_1 such that $A_2 - x_2$ is connected (again by condition (5)). Continue on this way to construct sequences $i_1, i_2, \dots, A_1, A_2, \dots, P_1, P_2, P_3, \dots$ and x_1, x_2, \dots

Since G is finite, there is a smallest t such that $A_{t+1} - x_t = A_s - x_s$ for some $s < t$. Put $B = A_s - x_s$. By condition (6) of r_{i_s} -permissibility, we have $z \in N_B(x_t) \cap N_B(x_s) \cap A_s^r$.

We now modify P_s to contradict the minimality of $f(P)$. Consider the set $X = \{x_{s+1}, x_{s+2}, \dots, x_{t-1}\} \cap V_{s,i_s}$. For $x_j \in X$, since $x_j \in A_j^r$ and x_j is not adjacent to x_{j-1} we see that $d_{G[V_{s,i_s}]}(x_j) \geq r_{i_s}$. Similarly, $d_{G[V_{s,i_t}]}(x_t) \geq r_{i_t}$. Also, by the minimality of t , X induces an independent set in G . Thus we may move all elements of X out of V_{s,i_s} to get a new partition $P^* = (V_{*,1}, \dots, V_{*,k})$ with $f(P^*) = f(P)$. Since x_t is adjacent to exactly r_{i_s} vertices in V_{t+1,i_s} and the only possible neighbors of x_t that were moved out of V_{s,i_s} between steps s and $t+1$ are the elements of X , we see that $d_{G[V_{*,i_s}]}(x_t) = r_{i_s}$. Since $d_{G[V_{*,i_t}]}(x_t) \geq r_{i_t}$ we can move x_t from V_{*,i_t} to V_{*,i_s} to get a new partition $P^{**} = (V_{**,1}, \dots, V_{**,k})$ with $f(P^{**}) = f(P^*)$. Now, remember our vertex $z \in V_{**,i_s}$. Since z is adjacent to x_t we have $d_{G[V_{**,i_s}]}(z) \geq r_{i_s} + 1$. Thus we may move z out of V_{**,i_s} to get a new partition P^{***} with $f(P^{***}) < f(P^{**}) = f(P)$. This contradicts the minimality of $f(P)$. \square

Question. Are there any other interesting r -permissible collections?

Question. The definition of r -permissibility can be weakened in various ways and the proof will still go through. Does this yield anything interesting?

Acknowledgments

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References

- [1] O.V. Borodin and A.V. Kostochka. On an upper bound of a graph's chromatic number, depending on the graph's degree and density. *Journal of Combinatorial Theory, Series B*, **23**, 1977, 247-250.
- [2] R. L. Brooks. On colouring the nodes of a network. *Math. Proc. Cambridge Philos. Soc.*, **37**, 1941, 194-197.
- [3] P. A. Catlin. A bound on the chromatic number of a graph. *Discrete Math*, **22**, 1978, 81-83.

- [4] P. A. Catlin. Another bound on the chromatic number of a graph. *Discrete Math*, **24**, 1978, 1-6.
- [5] A. V. Kostochka. A modification of a Catlin's algorithm. *Methods and Programs of Solutions Optimization Problems on Graphs and Networks*, **2**, 1982, 75-79 (in Russian).
- [6] J. Lawrence. Covering the vertex set of a graph with subgraphs of smaller degree. *Discrete Math*, **21**, 1978, 61-68.
- [7] L. Lovasz. On decomposition of graphs. *Studia Sci. Math Hungar.*, **1**, 1966, 237-238.